THE EVOLUTION OF A SOLITARY WAVE IN A NONHOMOGENEOUS MEDIUM

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Using the example of surface waves in a heavy liquid, the article discusses the propagation of a solitary wave in a nonhomogeneous medium. An analysis is made of processes of the decomposition of a wave into solitary waves, as a function of differences in the depth.

1. Solitary waves, as a specific type of nonlinear wave motion in dispersion media, have been studied intensively in connection with problems in hydraulics, plasma physics, and electrodynamics. As is well known [1, 2], an arbitrary perturbation, over the course of time, decomposes into a number of solitary waves, whose number and amplitude may be found from the laws of conservation. In connection with the problem of the existence and stability of nonlinear stationary waves, the evolution of a solitary wave in a nonhomogeneous medium is of interest. In the present article, this problem is considered for waves of variable depth, for which a relatively simple experimental verification of the theoretical data is possible. A number of problems with respect to the propagation of a solitary wave in a nonhomogeneous medium have already been discussed in the literature, for example, the evolution of an isolated wave has been studied in a linear approximation [3], and the problem of the "adiabatic" restructuring of a solitary wave, with a sufficiently slow change of the depth, has been solved in a nonlinear statement [4, 5]. Numerical solutions are known to the nonlinear problem of a solitary wave running against the shore [6, 7], and there has been experimental work on this subject [8, 9].

The present article discusses the evolution of a solitary wave in a nonhomogeneous zone of finite dimensions; the asymptotic form of the wave motion at $t \to \infty$ is found.

2. As is well known, the behavior of a wave on the surface of a liquid of variable depth is mainly determined by three independent parameters: the nonlinearity (the Mach number $M = u/\sqrt{gh}$; the dispersion $D = h^2/\lambda^2$; and the nonhomogeneity $N = \lambda/L$, where u is the velocity of the particles of the liquid, g is the acceleration due to gravity, h is the depth of the liquid, λ is the wave length, and L is the characteristic dimension of the nonhomogeneity. Depending on the relationships between these parameters, different types of wave motion may develop. Limiting ourselves in what follows to the case of a solitary wave of small amplitude ($M, D \ll 1$), depending on the value of the parameter of the nonhomogeneity, we distinguish five characteristic regions in which the wave processes are different: region I, where $N \gg 1$; region II, where $N \sim 1$; region III, where $M, D \ll N \ll 1$; region IV, where $N \sim M, D$; and region V, where $N \ll M, D$. In regions I-III, the nonhomogeneity is sharp ($N \gg M, D$), so that the nonlinear and dispersion effects cannot act during the time required for a pulse to pass through a zone of variable depth; in these regions, the solution is described by the formulas of the linear theory. In region V the nonhomogeneity is sufficiently smooth so that the wave, locally, remains stationary, and its amplitude and duration vary "adiabatically." Another classification of wave processes may also be made. In regions III-V the nonhomogeneity is smooth, $N \ll 1$, and the reflected wave may be neglected. In region I, $N \gg 1$, on the contrary, the nonhomogeneity may be approximated by a step, which simplifies the problem considerably. Of course, the above separation into five regions is of an arbitrary character, since the exact boundaries of the regions cannot be defined. We shall also assume that the overall drop in the depth is relatively small, so that the Korteveg—de Vries approxima-

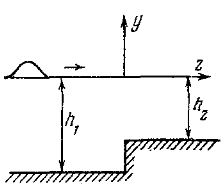

Fig. 1

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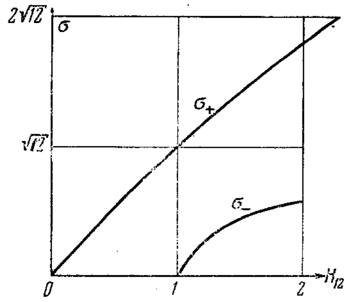

Fig. 2

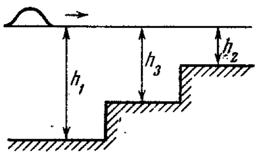

Fig. 3

tion is valid on both sides of the nonhomogeneous zone [1, 2]. In what follows, we shall consider the wave motions in each of the above regions.

3. Region I. In this region the wave processes are studied on the basis of the formulas of the linear theory; the depth varies jumpwise (Fig. 1). This problem is easily solved (see, for example [10]) and, for a given falling wave

$$u_0(t, z) = A_0 \,\mathrm{sch}^2\left[\frac{t - z/(\sqrt{gh_1} + A_0/3)}{T_0}\right], \quad T_0^2 = 2h_1^{3/2}/A_0 g^{1/2} \tag{3.1}$$

for the forward u_+ and reflected u_- velocities we find the following expressions:

$$u_+(t,\ 0) = PA_0 \,\mathrm{sch}^2 t/T_0, \quad u_-(t,\ 0) = RA_0 \,\mathrm{sch}^2 t/T_0$$
$$P = \frac{2\sqrt{H_{12}}}{\sqrt{H_{12}} + 1}, \quad R = \frac{\sqrt{H_{12}} - 1}{\sqrt{H_{12}} + 1} \quad \left(H_{12} = \frac{h_1}{h_2}\right) \tag{3.2}$$

Thus, the forward and reflected pulses have a "solitary-type" form; however, their duration differs from the durations of stationary solitary waves, and they have the amplitudes PA_0 and RA_0, respectively. As a result of this, the pulses start to form into stationary waves. As is well known, the evolution of an initial perturbation is determined by the value of the parameter of similarity σ [1]

$$\sigma = T\sqrt{6Ag^{1/2}h^{-3/2}} \tag{3.3}$$

(for a solitary wave $\sigma_c = \sqrt{12}$). Using (3.1) and (3.2) we obtain for the forward and reflected waves

$$\sigma_+ = \sqrt{12}\sqrt{2H_{12}^2/(H_{12}^{1/2} + 1)}, \quad \sigma_- = \sqrt{12}\sqrt{(H_{12}^{1/2} - 1)/(H_{12}^{1/2} + 1)} \tag{3.4}$$

Figure 2 gives the dependences of σ_+ and σ_- on the drop in the depth H_{12}. If the wave is transformed on a slope with $H_{12} > 1$, the reflected wave is formed only into one solitary wave (in this case there is the possibility of the formation of an oscillating wake behind the pulse [1]), since $\sigma_- < \sqrt{12}$. The forward wave separates into solitary waves, whose number is determined by the laws of conservation [1]. Thus, with $\sigma_+ < 2\sigma_c$ ($H_{12} < 2.25$), two solitary waves are formed with the amplitudes

$$\sqrt{A_{1,2}/PA_0} = \frac{\sigma_+}{2\sigma_c}\left[1 \pm \sqrt{1 - \frac{4\sigma_+^2/\sigma_c^2 - 1}{3\sigma_+^2/\sigma_c^2}}\right] \tag{3.5}$$

With $H_{12} > 2.25$, the forward wave separates into three solitary waves, etc. If $H_{12} < 1$, the forward wave is formed into a single solitary wave, while the reflected wave does not tend toward a stationary state at all, since, for it, $\int u_- dt < 0$; in this case a wave packet with a high-frequency filling will be propagated toward the side $z < 0$ (see, for example [11]). In what follows, we shall consider the falling of a wave into a zone with $H_{12} > 1$.

4. Region II ($N \sim 1$). In this region it is also possible to neglect the influence of nonlinear and dispersion effects with the passage of a pulse through a zone of variable depth; however, the actual relief of the bottom must be taken into account. The mathematical problem comes down to the solution of linear differential equations, but their integrals are unknown if the depth varies arbitrarily. For certain functions h (z) (for example, a linear slope) the solution is expressed in terms of special functions and the problem of the passage of a pulse through a nonhomogeneous zone is solved completely. Without going through the required calculations for these special cases, we limit ourselves to a few evaluations which bring out the effect of the finite nature of the dimensions of the transition region. It is clear that the presence of a finite length of the nonhomogeneous zone leads to an increase in the coefficient of the passage of the wave and, consequently, to

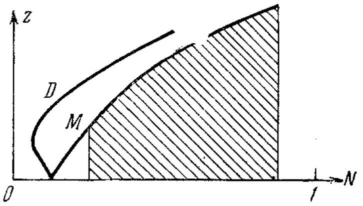

Fig. 4

a large value of σ_+. For example, considering a transformation on a slope, approximated by two steps (Fig. 3), we obtain

$$\sigma_+^{(2)} = \sqrt{12}\sqrt{4H_{12}^2/(H_{13}^{1/2}+1)(H_{32}^{1/2}+1)} \quad (H_{13}=h_1/h_3,\ H_{32}=h_3/h_2) \tag{4.1}$$

It is easily seen that $\sigma_+^{(2)} > \sigma_+$ with an identical overall drop H_{12} and, consequently, that with the presence of two steps there is the possibility of the formation of a larger number of solitary waves than with the transformation of a wave on one step. A maximal value of the function $\sigma_+^{(2)}$ is attained at $h_3 = H_{12}^{1/2}$.

We must emphasize once again that in regions I–III the problem of the evolution of a solitary wave is divided into two simple problems: the transformation of a solitary wave in a homogeneous zone (the linear approximation), and the decomposition of the initial pulse in a homogeneous zone.

5. Regions III–V ($N \ll 1$). With a smooth nonhomogeneity, the reflected wave can be neglected, and the hydrodynamic equations can be brought down to the "nonhomogeneous" Korteveg–de Vries equation [4]

$$gh\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial \tau} + \frac{h^{5/2}}{6g^{1/2}}\frac{\partial^3 u}{\partial \tau^3} + \frac{3}{4}gu\frac{\partial h}{\partial z} = 0, \quad \tau = \int\frac{dz}{\sqrt{gh}} - t \tag{5.1}$$

If the parameters of the nonlinearity, dispersion, and nonhomogeneity are of the same order of magnitude, it is not possible to obtain a solution of Eq. (5.1). However, expressions for the moments of the velocity can be obtained in explicit form [1]. Thus, integrating (5.1) with respect to τ, for the first moment we easily obtain

$$h^{3/4}(z)\int_{-\infty}^{\infty} u(z,t)\,dt = h_1^{3/4}\int_{-\infty}^{\infty} u_0\,dt \tag{5.2}$$

Multiplying Eq. (5.2) by u, and carrying out analogous computations, for the second moment we obtain

$$h^{3/2}(z)\int_{-\infty}^{\infty} u^2(z,t)\,dt = h_1^{3/2}\int_{-\infty}^{\infty} u_0^2\,dt \tag{5.3}$$

It is easily seen that the second moment coincides with the mean value of the energy flux [4]. If the dispersion term $(\partial^3 u/\partial\tau^3)$ in (5.1) cannot be neglected, it is impossible to obtain expressions in explicit form for moments of higher order. In many cases, a knowledge of even two moments makes it possible to solve the problem of the decomposition of a wave into two solitaty waves; at the same time an exact solution in the nonhomogeneous region may be rather complicated. We shall consider each region separately.

6. Region III ($M, D \ll N \ll 1$). In this case, in Eq. (5.1) we can neglect the nonlinear and dispersion terms (since, as has already been pointed out, the linear approximation is valid in regions I–III), and the equation is easily integrated:

$$u(z,\tau) = A_0[h_1/h(z)]^{3/4}\,\mathrm{sch}^2\,\tau/T_0 \tag{6.1}$$

As is evident from (6.1), the amplitude of the wave increases, while the duration remains unchanged. A knowledge of the solution of (6.1) permits using the laws of conservation to solve the problem of the subsequent transformation of a wave in a zone with a constant depth ($h=h_2$). Omitting the computations, we obtain the result that the wave is divided into two solitary waves with the amplitudes

$$\sqrt{A_{1,2}/A_0} = {}^1/_2\,H_{12}^{9/2}\{1 \pm \sqrt{1 - {}^4/_3\,[1 - H_{12}^{-9/4}]}\} \tag{6.2}$$

if $H_{12} < 1.85$. With large drops, the wave is divided into three solitary waves, etc. For this case, the parameter of similarity is equal to

$$\sigma = \sqrt{12}H_{12}^{9/8} \tag{6.3}$$

We note that, in a weakly nonhomogeneous medium ($N \ll 1$), the parameter σ is greater than in the case of a jumpwise change in the depth ($N \gg 1$). This is connected with the fact that there is no reflection, as a result of which the wave which has been transformed through a nonhomogeneous zone "contains more solitary waves within itself."

A rise in the value of the amplitude of the wave leads to a situation in which nonlinear effects start to have an influence. Since in this case, the dispersion decreases ($D \approx h/gT_0^2$), the profile of the wave starts to become distorted. Under these conditions, the solution of Eq. (5.1) is described by a quasi-simple wave [4]

$$u(z,\tau) = A_0\,[h_1/h(z)]^{3/4}\,\mathrm{sch}^2\left\{T_0^{-1}\left[\tau - uh^{3/4}g\int_0^z h^{-7/4}dz'\right]\right\} \tag{6.4}$$

Although the profile of the wave is given implicitly by formula (6.4), its moments can easily be found, as before, using formulas (5.2) and (5.3). Therefore, in a zone with $h = h_2$, the wave divides into two solitary waves, in accordance with (6.2). With $H_{12} > 1.85$, the wave is transformed into a sequence of three solitary waves, etc. We note that the calculation of moments of any given order, required for solution of the problem of the decomposition of the wave into n solitary waves, presents no difficulties. Neglecting the dispersion term in (5.1) we easily find

$$h^{3n/4}\int_{-\infty}^{\infty} u^n dt = h_1^{3n/4}\int_{-\infty}^{\infty} u_0{}^n dt \quad (n = 1, 2, \ldots) \tag{6.5}$$

(With the presence of dispersion, only the first two moments of (5.2) and (5.3) can be found in explicit form.)

Thus, in region III, there is both the wave in the nonhomogeneous form, and its asymptotic form at $t \to \infty$.

The sole limitation on the drop in the depth (or, what is the same thing, on the length of the nonhomogeneous zone) is connected with the possibility of the collapse of the wave before it comes out into a region with a constant depth. For the solution of (6.4), the condition for the breaking of a wave has the form [4]

$$\int_0^z h^{-7/4}(z)\,dz = \frac{3\sqrt{3}}{4}\,gT_0A_0^{-1}h_1^{-3/4} \tag{6.6}$$

For a constant slope of the bottom $\partial h/\partial z = -h_1/L$; from this we obtain

$$H_{12} = 1 + \frac{9\sqrt{3}}{16}\,gT_0h_1L^{-1}A_0^{-1} \tag{6.7}$$

At depths corresponding to condition (6.7), the dispersion terms become considerable, which limits the steepness of the wave front (see Sec. 8). Thus, although the parameters of the wave are at first satisfied in region III, later on it is necessary to take account of the rising nonlinearity and of the analysis made above, which is valid in the region shown schematically in Fig. 4 (hatched region). With large values of z, the parameters of the wave correspond to region IV.

7. Region V. In this region, the depth varies sufficiently slowly so that, locally, the wave may be regarded as stationary. In this case, a solitary wave is "adiabatically" reorganized, maintaining its form unchanged. This case has been discussed in [4, 5], in which the result is obtained that the amplitude of a solitary wave varies as a function of the depth in the following manner:

$$A = A_0\,[h_1/h(z)]^{3/2} \tag{7.1}$$

The amplitude of a solitary wave rises more rapidly than the amplitude of a quasi-simple wave (6.4). This latter fact is connected with a decrease of the time (in a Riemann wave, the temporal duration does not vary) required for regeneration of the quasi-stationary form of the pulse. However, as is easily shown [see (5.2)], a solution in the form of a solitary wave with a variable amplitude does not correspond to the equation for the first moment. This is bound up with the following fact. Since the extension of a solitary wave is infinite, it is clear that it makes no sense to speak of the stationary nature of the whole pulse in the nonhomogeneous zone. However, the part of the pulse in which is concentrated, for example, 99% of the whole energy, is located locally in a region with a constant depth, and is quasi-stationary (we can even identify it with a solitary wave). Therefore, the failure to satisfy (5.2) can be attributed to "unsteady-state wakes," which also make a contribution to the integral $\int u dt$. Thus, during its propagation, the pulse retains its solitary-type form, radiating the "excess" part of the value of $\int u dt$ equal to $(H_{12}{}^{3/4} - 1)\int u_0 dt$ into the wake.* We emphasize that the condition for an adiabatic change in a solitary wave is $L \gg \lambda/M$, and not only $L \gg \lambda$; the latter is the condition for the absence of a reflected wave. Since the Mach number rises with propagation, the condition of the adiabatic approximation is satisfied with greater accuracy. This is connected with a decrease in the duration of a solitary wave with a growth in its amplitude.

*An analogous situation exists also for dissipative media in which, with small losses, a solitary wave retains its form independently of the value of $\int u dt$ [12, 13].

8. Region IV. In region IV, the parameters of nonlinearity, dispersion, and nonhomogeneity are of the same order of magnitude, and this case is intermediate between the cases discussed in Secs. 6 and 7. The effect of the dispersion leads to a situation in which the duration of the wave will partially "follow" behind the change in its amplitude; however, not to such a degree as with the adiabatic change of a solitary wave. In addition, there is an increase in the radiation of a solitary wave into the wake. This leads to a situation in which the pulse becomes broader compared to a solitary wave with the same amplitude, and the wakes have a finite energy. As a result, the amplitude of a pulse rises in accordance with a law of the type $[h_1/h(z)]^{\alpha}$, where $3/4 < \alpha < 3/2$. Using the laws of conservation (5.2) and (5.3), it is impossible to give an unambiguous answer to the problem of the further evolution of a pulse. It is clear that, as a result of the large amount of radiation of the energy of a solitary wave into the wake, the drop in the depth required for the formation of two solitary waves rises from 1.85 (region III) to infinity (region V). The complete solution of the problem of the transformation of a wave is bound up with the solution of a nonlinear equation with variable coefficients and, at the present time, is unknown. We note only that, when the amplitude of the wave rises to such a point that $M \gg N$, the solitary wave part of the pulse varies adiabatically, in accordance with the formula (7.1).

9. Let us compare the data obtained with the existing results. The formation of solitary waves after the passage of a wave through a nonhomogeneous zone has been observed under natural conditions [14]. In [7], an electronic computer was used for a numerical solution of the problem of the passage of a solitary wave over a sloping shore with an overall drop $H_{12} = 2$. In accordance with the classification adopted, the parameters of a solitary wave were referred to three regions, and the numerical solution confirmed the nonlinear deformation of the wave in accordance with (6.4). After its exit into a zone with a constant depth, the pulse decomposed into three solitary waves, which corresponds to the theoretical value at $H_{12} > 1.85$. A nonlinear distortion of the wave profile, corresponding to the conditions of region III, has also been observed in numerical calculations [6].

Several articles have discussed the transformation of a wave over a slope which does not come out into a zone of constant depth; in this case, the parameters M, D, N corresponded to region IV [6, 8, 9]. In these cases, the degree of increase of the amplitude in three dimensions increased as a function of the Mach number; for a smaller slope of the bottom, the amplitude of the wave was greater. The lack of complete information on wave processes in a nonhomogeneous zone does not permit a more detailed comparison between theoretical and experimental data.

The processes considered above also take place in magnetosonic waves in cosmic plasma, with a variable magnetic field.

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